

Projection methods for ill-posed problems revisited

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Abstract

The discretization of least-squares problems for linear ill-posed operator equations in Hilbert spaces is considered. The main subject of this article concerns conditions for convergence of the associated discretized minimum-norm least-squares solution to the exact solution using exact attainable data. The two cases of global convergence (convergence for all exact solution) or local convergence (convergence for a specific exact solution) are investigated. We review the existing results and prove new equivalent condition when the discretized solution always converges to the exact solution. An important tool is to recognize the discrete solution operator as oblique projection. Hence, global convergence can be characterized by certain subspaces having uniformly bounded angles. We furthermore derive practically useful conditions when this holds and put them into the context of known results. For local convergence we generalize results on the characterization of weak or strong convergence and state some new sufficient conditions. We furthermore provide an example of a bounded sequence of discretized solutions which does not converge at all, not even weakly.

1 Introduction

We study the role of discretization in the use of solving ill-posed linear operator equations in Hilbert spaces. Consider an ill-posed problem in Hilbert spaces

$$Ax = y, \quad (1)$$

where $A : X \rightarrow Y$ is continuous and equation (1) for solving x from given data y is ill-posed. In the following, $N(A)$ and $R(A)$ denote the nullspace and the range of an operator A , respectively. By A^\dagger we denote the pseudoinverse of A ; cf., e.g., [7]. We symbolize norm-convergence by \rightarrow and weak convergence by \rightharpoonup . We denote the weak limit by the symbol wlim , and, for a closed subspace Z , Π_Z denotes the associated orthogonal projector onto Z . In the following, we assume (unless specified otherwise) the attainable case for problem (1), i.e., that y is in $R(A)$. In this case, we can set y being the image of an element in $N(A)^\perp$.

$$Ax^\dagger = y, \quad x^\dagger \in N(A)^\perp.$$

It is the unique element x^\dagger , which we want to reconstruct from given data y .

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We are interested in projection methods acting as a regularization, i.e., in approximating the pseudoinverse of A by solving discrete least-squares problems related to (1). For this task we introduce discretizations in the spaces X and Y . Precisely, we assume given an increasing sequence of finite-dimensional spaces $X_n \subset X$ and $Y_m \subset Y$, $n, m \in \mathbb{N}$, with the property

$$X_n \subset X_{n+1}, \quad \overline{\bigcup_n X_n} = X, \quad Y_m \subset Y_{m+1}, \quad \overline{\bigcup_m Y_m} = Y. \quad (2)$$

For the discretization spaces we always denote the associated orthogonal projector onto X_n by $P_n := \Pi_{X_n}$ and onto Y_m by $Q_m := \Pi_{Y_m}$

$$P_n : X \rightarrow X_n, \quad Q_m : Y \rightarrow Y_m.$$

The discretization of (1) by a general projection method involves the operator

$$A_{n,m} := Q_m A P_n, \quad (3)$$

and we define the associated solutions by (assuming attainability)

$$x_{n,m} := A_{n,m}^\dagger y = A_{n,m}^\dagger A x^\dagger. \quad (4)$$

It is well-known that $x_{n,m}$ is the unique solution of minimum norm under all least-squares solutions of the projected problem, i.e.,

$$x_{n,m} = \operatorname{argmin}_{x \in X_n} \|Q_m A x - y\|^2 = \operatorname{argmin}_{x \in X_n} \|Q_m A x - Q_m y\|^2, \quad \text{and} \\ x_{n,m} \in N(A_{n,m})^\perp.$$

It follows that $A_{n,m}^\dagger = P_n A_{n,m}^\dagger Q_m$. The general projection method (4) embraces two special well-known methods: if we put formally $m = \infty$, and hence $Q_m = I$, we obtain the projected least-squares method involving

$$x_n := A_n^\dagger A x^\dagger, \quad A_n = A P_n. \quad (5)$$

Conversely if we set $n = \infty$ and formally put $n = \infty$, we obtain the dual least-squares method,

$$x_{\infty,m} := A_{\infty,m}^\dagger A x^\dagger, \quad A_{\infty,m} = Q_m A.$$

We distinguish these important special cases by labeling them with only one index for the first method and by the index ∞, m for the second one. However, the dual least-squares method is not so much of interest for this paper (although it is of practical importance) as it always leads to a convergent method.

It is clear that $A_{n,m}^\dagger$ is a bounded operator and hence $x_{n,m}$ can be computed in a stable way. Moreover, the usual rules for adjoints and inverses hold: $(A_{n,m}^\dagger)^* = (A_{n,m}^*)^\dagger$.

The immediate question that arises from this setup is, if $x_{n,m}$ in (4) converges to x^\dagger as $n, m \rightarrow \infty$, in what sense does this convergence happen, and for which x^\dagger does this hold.

More precisely, we study two different subjects:

- **Local convergence.** Fix x^\dagger . Find conditions such that

$$x_{n,m} \rightharpoonup x^\dagger \quad \text{as } n, m \rightarrow \infty,$$

or

$$x_{n,m} \rightarrow x^\dagger \quad \text{as } n, m \rightarrow \infty.$$

- **Global convergence.** Find conditions such that

$$x_{n,m} \rightarrow x^\dagger \quad \text{as } n, m \rightarrow \infty \quad \forall x^\dagger \in N(A)^\perp.$$

The second issue concerns convergence not only for one fixed x^\dagger but for all $x^\dagger \in N(A)^\perp$.

Both questions are relevant for the general projection method (4) and the projected least-squares method (5). Note that the distinction between weak and strong convergence is not relevant for global convergence because the corresponding conditions are identical [9].

Of course, these questions have been discussed and partly answered in literature, but often only for the projected least-squares method or even with further restriction like injective operators; see Section 2.1 for a review. It is observed that many authors in different articles use different conditions to prove convergence of a specific scheme, for instance, (6) or (7) below. The relation between different conditions in different papers is not always obvious. It is one of the purposes of this paper to clarify this situation and to unify the convergence conditions at best, to generalize known results to the general projection case using the operator (3) and hereby avoiding unnecessary assumptions like injectivity.

Let us mention that the convergence of $x_{n,m}$ to x^\dagger is the most important requirement for the projection methods discussed here to act as regularization. The second one, the stability of the regularization, is automatically satisfied since we are dealing with finite-dimensional problems. Indeed, if convergence of $x_{n,m}$ to x^\dagger is verified, it is not difficult to find error estimates for noisy data as well and with appropriate parameter choice rules (where the index of the approximation spaces n, m act as “regularization parameter”), convergence of $x_{n,m}$ to x^\dagger can be proven even for the case of noisy data. We do not dwell further on this matter since it can be treated by standard methods; for results on the noisy case or also nonlinear problems, see, e.g., [1, 3, 8, 11, 13, 14, 15, 16, 23]; for combination with regularization, see e.g., [25, 30, 31, 34]. For results with focus on the analysis of specific advanced method of choosing the discretization spaces (like adaptivity or multilevel-type), we refer to [17, 18, 22].

This paper is organized as follows: in Section 2, we review existing convergence results and prove some important lemmas. In Section 3 we provide new conditions for global or local convergence and relate them to results in literature. In Section 4 we state a nontrivial example of a non-convergent sequence $x_{n,m}$ which is bounded. We summarize with a conclusion in Section 5.

2 Known and preliminary results

In this section we give an extensive literature review of known results related to the questions raised in the previous section. Moreover, we present some lemmas needed later for the convergence analysis.

2.1 A review of known results

The question of local or global convergence has, of course, been addressed in several articles. However, as stated above, quite often only injective operators, i.e., $N(A) = \emptyset$, or the case of projected least-squares problems, i.e., $Q_m = I$,

have been addressed. Moreover, although those results are useful, they are not always completely sharp.

Before we come to the positive results, we remind of a well-known negative result of non-convergence. The following statement is the famous counterexample of Seidman [32] for the projected least-squares problem.

Example 1 (Seidman). *There exists a compact injective linear operator A and a x^\dagger such that x_n as given by (5) is a unbounded sequence. Thus, in particular, we have non-convergence $x_n \not\rightarrow x^\dagger$. Moreover there also exists A, x^\dagger as before such that x_n is bounded but $x_n \not\rightarrow x^\dagger$.*

The operator used for this example is a diagonal operator in the l^2 -sequence space with a rank-1 perturbation:

$$A : l^2 \rightarrow l^2 \quad A = \text{diag}(\gamma) + \beta \otimes e_1,$$

where γ and β are appropriate sequences and e_1 is the sequence with all 0 except at the first position, where it is 1. By an appropriate (constructive) choice of x^\dagger and β, γ , the unboundedness of x_n can be shown; see [32] or [7]. The last statement in this theorem of a bounded (strongly-) non-convergence sequence is stated in [32] but not explicitly proven.

Concerning the question of finding conditions for global convergence, the problem is well-studied. The following result is proven by Nashed [26], (for A_n), see also [27], in [20, Theorem 3.7] for A_n being injective, and for the general case with $A_{n,m}$ by Du [5] (see also [6]). It gives a necessary and sufficient condition for global convergence.

Theorem 1.

$$x_{n,m} \rightarrow x^\dagger \quad \text{as } m, n \rightarrow \infty \quad \forall x^\dagger \in N(A)^\perp,$$

if and only if there exists a constant C such that

$$\sup_{n,m} \|A_{n,m}^\dagger A\| \leq C. \quad (6)$$

Below, we will also reprove the corresponding result (Theorem 15) and, in particular, study characterizations of the uniform boundedness condition (6); see Theorems 17 and 18.

Note that in [5, Theorem 2.6], Theorem 1 has been generalized to the case of nonattainable data, i.e., when $y = Ax^\dagger + R(A)^\dagger$. In this case the necessary and sufficient conditions for x_n being strongly (weakly) convergent to $A^\dagger y$ is (6) and $A_n^* Q_n y \rightarrow 0 \Rightarrow A_n^\dagger Q_n y \rightarrow (\rightarrow) 0$.

It follows immediately from Theorem 1 for the dual projection case, i.e., $A_{n,m} = A_{\infty,m}$, by $A_{\infty,m}^\dagger = A_{\infty,m}^\dagger Q_m$, that condition (6) is always satisfied, i.e., this method always globally converges. This is well-known and has been shown, e.g., in [7].

A widely used sufficient condition for uniform boundedness and hence global convergence of x_n has been presented by Natterer [27] using a result by Nitsche [29].

Theorem 2 (Natterer). *Let A be injective. Suppose that there exists a constant C such that for all $x^\dagger \in X$ there exists a $u_n \in X_n$:*

$$\|x^\dagger - u_n\| + \|A_n^\dagger\| \|A(x^\dagger - u_n)\| \leq C \|x^\dagger\|. \quad (7)$$

Then $x_n \rightarrow x^\dagger$ as $n \rightarrow \infty$ for all $x^\dagger \in X$.

In this theorem,

$$\|A_n^\dagger\| = \sup_{\|A_n x_n\|=1, x_n \in X_n} \|x_n\| = \sigma_{\min}^{-1}(AP_n).$$

It is not difficult to verify that (7) implies (6). We will generalize this result by giving a condition resembling (7) which is equivalent to (6) and hence yields global convergence in the general case (including non-injective operators and for the general projection case); see below Proposition 19.

Furthermore a quite general condition has been proposed by Vainikko and Hämarik [35] (see also [11, 14, 15] and [12] and the references therein).

Theorem 3 (Vainikko and Hämarik). *Suppose that $N(Q_m AP_n A^*) = \{0\}$. If there is a constants C such that*

$$\|A^* Q_m AP_n z\| \leq C \|P_n A^* Q_m AP_n z\| \quad \forall z \in X_n, \quad (8)$$

then $x_{n,m} \rightarrow x^\dagger$ for all $x^\dagger \in N(A)^\perp$.

Further results, e.g., on appropriate parameter choice rules, are proven in [35] as well. We will show below (Theorem 18) that (8) is actually equivalent to (6).

A simple condition involving the product of the ill-posedness and approximation rate has been used by several authors (e.g., [19, 24])

Theorem 4. *If $\|A(I - P_n)\| \|A_{n,m}^\dagger\| < \infty$, then $x_{n,m} \rightarrow x^\dagger$ as $m, n \rightarrow \infty$ for all $x^\dagger \in N(A)^\perp$. If for a specific x^\dagger , $\lim_{m,n \rightarrow \infty} \|A(I - P_n)x^\dagger\| \|A_{n,m}^\dagger\| = 0$, then $x_{n,m} \rightarrow x^\dagger$ as $m, n \rightarrow \infty$.*

The previous results are ones that hold uniformly for all x^\dagger (except for the very last one), and convergence for all $x^\dagger \in N(A)^\perp$ is obtained. However, it is of high interest to study conditions for convergence for one specific x^\dagger , when we do not care about global convergence. There are some statements concerning local convergence in literature.

In a quite general situation, necessary and sufficient conditions for local convergence have been established by Groetsch and Neubauer [7, 9].

Theorem 5 (Groetsch and Neubauer, also Du). *We have the following local convergence conditions for strong convergence:*

$$x_n \rightarrow x^\dagger \iff \limsup_n \|x_n\| \leq \|x^\dagger\|.$$

Moreover, suppose that

$$\overline{\bigcup_n (N(A) \cap X_n)} = N(A). \quad (9)$$

Then we have the following local convergence conditions for weak convergence:

$$x_n \rightharpoonup x^\dagger \iff \sup_n \|x_n\| < \infty. \quad (10)$$

Note that the part on weak convergence in Theorem 5, identity (10) is erroneously stated in [7, 9] without the space condition (9), as has been noted by Du [5]. The characterization of strong convergence is valid without (9) and was already stated in [9] (using the incomplete result for weak convergence). It has been rigorously proved by Du and Du [6, Remark 4.3]. We will extend Theorem 5 to general projection methods with $x_{n,m}$; see Theorem 25.

Besides the convergence result of Neubauer and Groetsch, a sufficient condition for strong local convergence without requiring information about x^\dagger has been stated by Luecke and Hickey [21].

Theorem 6 (Luecke, Hickey). *Suppose that*

$$\sup_n \|(A_n^\dagger)^* x_n\| < \infty, \quad (11)$$

then $x_n \rightarrow x^\dagger$.

This result is also proven in [7], where it is also explained that (11) is quite strong (and thus not a necessary condition for convergence) as it leads to a convergence rate of $x_n - x^\dagger$. In Proposition 27 we provide a similar result but by employing weaker conditions.

A subtle and important point is the space condition (9). In case of injective operators, of course, (9) holds true but in the general case not always, not even if $N(A)$ is finite-dimensional. (Think, for instance, of a discretization space X_n that is disjoint to $N(A)$.) We note that $(N(A) \cap X_n)$ is an increasing family of closed subspaces, thus the following identity holds, (cf., e.g., [10, Chpt. 1, § 12])

$$\overline{\bigcup_n (N(A) \cap X_n)}^\perp = \bigcap_n (N(A) \cap X_n)^\perp,$$

so that (9) is equivalent to

$$\bigcap_n (N(A) \cap X_n)^\perp = N(A)^\perp. \quad (12)$$

A recent preprint [6, Theorem 1.1] discusses equivalent conditions to (9) (respectively, (12)).

Theorem 7. *The condition (9) is equivalent to each of the following conditions, Here $\mathcal{G}(A)$ denotes the graph of an operator A .*

•

$$\forall x \in N(A) : \lim_{n \rightarrow \infty} \inf_{z_n \in N(A_n)} \|x - z_n\| \rightarrow 0,$$

•

$$\forall (x, y) \in \mathcal{G}(A^\dagger) : \lim_{n \rightarrow \infty} \inf_{(z_n, w_n) \in \mathcal{G}(A_n^\dagger)} \|(x, y) - (z_n, w_n)\| \rightarrow 0,$$

• *for all sequences y_n :*

$$\sup_n \|A_n^\dagger y_n\| < \infty \text{ and } y_n \rightharpoonup y \Rightarrow y \in \mathcal{D}(A^\dagger) \text{ and } A^\dagger y = A_n^\dagger y.$$

Moreover (6) is equivalent to the following two conditions holding simultaneously, (9) and $\text{gap}(R(A^*AP_n), R(A^\dagger AP_n)) < 1$, where the gap between two spaces M, N is defined as (see [6, Lemma 3.2]) $\text{gap}(M, N) = \|P_M - P_N\|$.

We will extend the second part of this result and we show that (6) can be equivalently be characterized as a certain angle (or gap) between subspaces (but not those in this theorem) to be smaller than one; cf. Lemma 16. Moreover, we also study local convergence also when (9) is not satisfied.

The subtle fact that boundedness of x_n is *not enough* for weak convergence and that an additional condition, e.g., like (9), is needed, is not very well-known. Du [5, Example 2.10] gave a counterexample of a sequence of $\|x_n\|$ being uniformly bounded but which does not converge weakly to x^\dagger .

Example 2 (Du). *There exists a linear operator A and a $x^\dagger \in N(A)^\perp$ such that*

$$\sup_n \|x_n\| < \infty,$$

but $x_n \not\rightharpoonup x^\dagger$. In this example, however, x_n converges strongly to some element $\neq x^\dagger$.

The operator in this counterexample is actually not ill-posed but a simple projection operator onto the complement of a one-dimensional subspace, $A = I - (., e)e$ with some appropriately chosen e . Failure of convergence happens because x_n converges (even strongly) but to the “wrong” solution.

Below in Theorem 28, we give a counterexample that is even more extreme: a situation like in the previous result, Example 2, but where the sequence x_n does *not* converge at all (not even weakly). This example has been devised by Neubauer [28].

Example 3 (Neubauer). *There exists a linear operator A and a $x^\dagger \in N(A)^\perp$ such that*

$$\sup_n \|x_n\| < \infty,$$

but x_n does not converge weakly.

Moreover, the sequence x_n has a subsequence, which converges weakly but with limit $u \neq x^\dagger$, and no weakly convergent subsequence has limit x^\dagger .

2.2 Preliminary lemmas

Since $x_{n,m}$ is always in $N(A_{n,m})^\perp$, it is important to study these spaces. We note that by discretization, $A_{n,m}$ is an operator with closed range. We have the well-known duality relations,

$$R(A_{n,m}^*) = N(A_{n,m})^\perp \quad \text{and} \quad N(A_{n,m}^*) = R(A_{n,m})^\perp, \quad (13)$$

and all these spaces are closed.

The following characterizations follow easily.

Lemma 8.

$$\begin{aligned} N(A_{n,m}) &= \{x \in X \mid x = w_n + q_n : w_n \in X_n \cap N(Q_m A), q_n \in X_n^\perp\} \\ N(A_{n,m})^\perp &= \{x \in X_n \cap (N(Q_m A) \cap X_n)^\perp\} \\ &= \{x \in X_n \mid \exists v_n : x = P_n A^* Q_m v_n\}, \end{aligned} \quad (14)$$

Proof. The first identity (14) follows easily by $x = P_n x + (I - P_n)x$. It is straightforward to proof that any $x \in X_n \cap (X_n \cap N(Q_m A))^\perp$ is in $N(A_{n,m})^\perp$. Conversely, by taking either w_n or q_n to be 0, it follows that any $x \in N(A_{n,m})^\perp$ must be in both X_n and $(X_n \cap N(Q_m A))^\perp$. The last identity is (13). \square

Note that these spaces are not necessarily nested. However, the following inclusions can be verified:

$$N(Q_m A) \cap X_n \subset N(Q_m A) \cap X_{n+1} \subset \dots \subset N(Q_m A), \quad (15)$$

$$N(Q_m A) \cap X_n \supset N(Q_{m+1} A) \cap X_n \supset \dots \supset N(A) \cap X_n, \quad (16)$$

$$N(A) \cap X_n = \bigcap_m (N(Q_m A) \cap X_n). \quad (17)$$

A simple consequence is that, for a fixed x , the norm of the projection $\|\Pi_{N(Q_m A) \cap X_n} x\|$, is increasing in m (for fixed n) and decreasing in n (for fixed m).

We now state some approximation results, namely that elements in $N(A)^\perp$, i.e., the space where x^\dagger lives, can be approximated arbitrary well by elements in the corresponding discrete space $N(A_{n,m})^\perp$.

Lemma 9. For all $x \in N(A)^\perp$,

$$\lim_{n,m \rightarrow \infty} \inf_{z \in R(A_{n,m}^*)} \|x - z\| = \lim_{n,m \rightarrow \infty} \inf_{z \in N(A_{n,m})^\perp} \|x - z\| = 0, \quad (18)$$

Proof. Let $x \in N(A)^\perp = \overline{R(A^*)} = \overline{R(A^* A)}$. Then for any $\epsilon > 0$ fixed, we can find a w_ϵ such that

$$\|x - A^* w_\epsilon\| \leq \epsilon.$$

Moreover by (2), with ϵ and w_ϵ as before, we can find a n_0 such that for all $n \geq n_0$ and $m \geq n_0$

$$\|P_n A^* w_\epsilon - A^* w_\epsilon\| \leq \epsilon, \quad \|Q_m w_\epsilon - w_\epsilon\| \leq \frac{\epsilon}{\|A^*\|},$$

thus,

$$\|x - P_n A^* Q_m w_\epsilon\| \leq \|x - P_n A^* w_\epsilon\| + \|P_n A^* (I - Q_m) w_\epsilon\| \leq 2\epsilon.$$

which yields (18). Note that $P_n A^* Q_m w_\epsilon$ is in $R(A_n^*) = N(A_n)^\perp$. \square

We remark that $N(A_{n,m})^\perp$ is not necessarily a subspace of $N(A)^\perp$ so that it is not correct to say that $N(A_{n,m})^\perp$ is dense in $N(A)^\perp$.

Furthermore, the operators $A^* A_{n,m}$ and $A_{n,m}^* A$ will play an important role in the subsequent analysis.

Lemma 10.

$$N(A_{n,m}) = N(A^* A_{n,m}) \quad R(A_{n,m}^* A) = R(A_{n,m}^*)$$

Proof. Clearly $N(A_{n,m}) \subset N(A^* A_{n,m}) = N(A^* A_{n,m})$. Conversely for $x \in N(A^* A_{n,m})$ we have $A^* Q_m A P_n x = 0$, and hence also $P_n A^* Q_m Q_m A P_n x = 0$, thus $x \in N(A_{n,m})$, which shows $N(A_{n,m}) = N(A^* A_{n,m})$, and by (13) the lemma follows. \square

Using $A^*A_{n,m}$ and $A^*A_{n,m}$, we have a characterization of the solution operator $A_{n,m}^\dagger A$ as a certain nonorthogonal projection operator.

Proposition 11. *For any $x \in X$ and any $n, m \in \mathbb{N}$, we have the unique decomposition*

$$x = v_{n,m} + u_{n,m} \quad v_{n,m} \in R(A_{n,m}^*A), u_{n,m} \in N(A_{n,m}^*A). \quad (19)$$

Moreover the mapping $x \rightarrow v_{n,m}$ is given by $A_{n,m}^\dagger A$, i.e.,

$$v_{n,m} = A_{n,m}^\dagger Ax.$$

For any $x \in X$ any $n, m \in \mathbb{N}$, we have the unique decomposition

$$x = \bar{v}_{n,m} + \bar{u}_{n,m} \quad \bar{v}_{n,m} \in R(A^*A_{n,m}), \bar{u}_{n,m} \in N(A^*A_{n,m}) \quad (20)$$

$$\begin{aligned} & \bar{v}_{n,m} \in R(A^*A_{n,m}), \\ & = \bar{v}_{n,m} + \bar{w}_{n,m} + \bar{q}_{n,m} \quad \bar{w}_{n,m} \in N(Q_m A) \cap X_n, \\ & \bar{q}_{n,m} \in X_n^\perp. \end{aligned} \quad (21)$$

Here, $\bar{v}_{n,m}, \bar{u}_{n,m}, \bar{w}_{n,m}, \bar{q}_{n,m}$ are uniquely determined. Moreover, the mapping $x \rightarrow \bar{v}_{n,m}$ is given by $A^*(A_{n,m}^*)^\dagger$, i.e.,

$$\bar{v}_n = A^*(A_{n,m}^*)^\dagger x,$$

and the mapping $x \rightarrow \bar{w}_{n,m}$ is given by

$$\bar{w}_{n,m} = \Pi_{N(Q_m A) \cap X_n} x. \quad (22)$$

Proof. Define $v_{n,m} = A_{n,m}^\dagger Ax$, then $v_{n,m} \in N(A_{n,m})^\perp = N(A^*A_{n,m})^\perp = R(A_{n,m}^*A)$. In particular, we have $v_{n,m} \in X_n$. Moreover v_n satisfies the normal equations

$$0 = P_n A^* Q_m (A P_n v_n - Ax) = P_n A^* Q_m A (v_n - x).$$

Thus $v_n - x \in N(A_{n,m}^*A)$ yielding the desired decomposition (19). Conversely, for any other decomposition as above, it follows that $v_n \in N(A_{n,m})^\perp$, and it satisfies the normal equations. By uniqueness of the minimal-norm least-squares solution, it follows that $v_n = A_{n,m}^\dagger Ax$. Thus the decomposition is unique. For the second part, define $z_{n,m} = (A_{n,m}^*)^\dagger x$, then $z_{n,m} \in N(A_{n,m}^*)^\perp = R(A_{n,m})$. In particular $z_{n,m} \in Y_m$. The normal equation implies that $Q_m A P_n A^* z_{n,m} - Q_m A P_n x = 0$, thus $\bar{v}_{n,m} - x = A^* z_{n,m} - x \in N(A_{n,m}) = N(A^*A_{n,m})$, which gives the decomposition. Any other decomposition of the form $x = A^*A_{n,m}p + N(A_{n,m})$ implies that $A_{n,m}p$ satisfies the same normal equation as $z_{n,m}$ and it clearly is in $R(A_{n,m}) = N(A_{n,m}^*)^\dagger$, thus by the uniqueness of the minimal-norm least-squares solution, we have $A_{n,m}p = (A_{n,m}^*)^\dagger x$. Hence, $A^*A_{n,m}p = A^*(A_{n,m}^*)^\dagger x$, which implies the unique decomposition (20). The decomposition of $\bar{u}_{n,m}$, into $\bar{w}_{n,m} + \bar{q}_{n,m}$ exists by the characterization of $N(A_{n,m})$ in (14) and is clearly unique since $\bar{w}_{n,m}$ is orthogonal to $\bar{q}_{n,m}$. Since $\bar{v}_{n,m} = A^*z_{n,m} = A^*Q_m z_{n,m}$, it follows that $\bar{v}_{n,m}$ is orthogonal to $N(Q_m A)$ and clearly $\bar{q}_{n,m}$ is orthogonal to X_n , hence applying $\Pi_{N(A) \cap X_n}$ to the decomposition gives the representation for $\bar{w}_{n,m}$. \square

Remark 1. This proposition will be used widely; in particular, we recognize that $x_{n,m} = A_{n,m}^\dagger A x^\dagger$ is the first element in (19) in the decomposition of x^\dagger and thus $x_{n,m}$ is the result of a nonorthogonal (oblique) projection applied to x^\dagger .

From (19), (20) we also obtain the nontrivial fact that

$$N(A_{n,m}^* A) \cap R(A_{n,m}^* A) = \emptyset, \quad \text{and} \quad N(A^* A_{n,m}) \cap R(A^* A_{n,m}) = \emptyset.$$

As a corollary we have a formula for $\Pi_{N(Q_m A) \cap X_n}$.

Corollary 12. For any $k \leq n$,

$$P_k \Pi_{N(Q_m A) \cap X_n} = P_k - A_k^* (A_{n,m}^*)^\dagger,$$

$$\Pi_{N(Q_m A) \cap X_n} P_k = P_k - A_{n,m}^\dagger A_k.$$

Proof. Applying P_k to (21) and using the fact that orthogonal projectors are selfadjoint and $(A_k^* (A_{n,m}^*)^\dagger)^* = A_{n,m}^\dagger A_k$ yields the result. \square

As another illustration of the usefulness of Proposition 11, we can prove a similar characterization of the space condition (9) as in Theorem 7.

Proposition 13. We have that (9) is satisfied if and only if

$$\left(\lim_{n,m \rightarrow \infty} \inf_{z_{n,m} \in R(A_{n,m}^*)} \|x - z_{n,m}\| = 0 \right) \implies x \in \overline{R(A^*)} \quad (23)$$

Proof. Let (23) hold and suppose that (9) does not hold. Then there exists a $x \neq 0$ and $x \in N(A)$ and $x \in \bigcap_n (N(A) \cap X_n)^\perp$. For such a x using (21), (22), and (17), it follows that for all n $\lim_{m \rightarrow \infty} \bar{w}_{n,m} = 0$. Thus, by Corollary 12 with $k = n$,

$$\lim_{m \rightarrow \infty} \|(P_n - A_n^* (A_{n,m}^*)^\dagger)x\| = \lim_{m \rightarrow \infty} \|(P_n - A_{n,m}^\dagger A_n)x\| = 0.$$

Taking n such that $\|x - P_n x\| \leq \epsilon$, we find a m such that

$$\|x - A_{n,m}^\dagger A_n x\| \leq \|x - P_n x\| + \|(P_n - A_{n,m}^\dagger A_n)x\| \leq 2\epsilon.$$

Thus by (23), since $A_{n,m}^\dagger A_n x \in R(A_{n,m}^*)$, it follows that $x \in \overline{R(A^*)} = N(A)^\perp$. Since $x \in N(A)$, we have a contradiction, thus (9) must hold. Conversely, if (9) holds, suppose that (23) does not hold. Then we have a $x \in N(A)$ and $z_{n,m} \in R(A_{n,m}^*)$ with $\|x - \Pi_{R(A_{n,m}^*)} x\| \rightarrow_{n,m} 0$. As $\Pi_{R(A_{n,m}^*)} = A_{n,m}^\dagger A_n = A_{n,m}^* (A_{n,m}^*)^\dagger$, we have that $x - A_{n,m}^* (A_{n,m}^*)^\dagger x \rightarrow_{n,m} 0$. Applying P_n to (21), it follows that $\bar{w}_{n,m} = P_n x - A_{n,m}^* (A_{n,m}^*)^\dagger x \rightarrow_{n,m} 0$; in particular $\lim_n \lim_m \bar{w}_{n,m} = \lim_n \Pi_{N(A) \cap X_n} x = 0$. By (15), $\Pi_{N(A) \cap X_n} x$ is increasing, hence $\Pi_{N(A) \cap X_n} x = 0$ for all n . In other words, $x \in \bigcap_n (N(A) \cap X_n)^\perp$. Using (9) implies that $x \in N(A)^\perp$, which is a contradiction to $x \in N(A)$. \square

In view of Lemma 9, we always have that

$$N(A)^\perp \subset \{x : \text{dist}(x, N(A_{n,m})^\perp) \rightarrow_{n,m} 0\}, \quad (24)$$

but according to (23), equality holds only if the space condition (9) holds. The corresponding result for the projected least-squares case using A_n (and more) has already been proven in [6].

3 Convergence results

We now study necessary and sufficient conditions for local and global convergence of $x_{n,m}$ to x^\dagger , thus extending the known results of the Section 2.1.

3.1 Conditions for global convergence

At first we consider convergence for all $x^\dagger \in N(A)^\perp$. We reprove the statement of Theorem 1 based on the following lemma.

Lemma 14. *For all $x^\dagger \in N(A)^\perp$ and $x_{n,m} = A_{n,m}^\dagger A x^\dagger$, we have*

$$\limsup_{n,m \rightarrow \infty} \|x_{n,m} - x^\dagger\| \leq \limsup_{n,m \rightarrow \infty} \|A_{n,m}^\dagger A(I - P_n)x^\dagger\|$$

Proof. Noting that $A_{n,m}^\dagger A_{n,m} = I - \Pi_{N(A_{n,m})} = \Pi_{N(A_{n,m})^\perp} = \Pi_{R(A_{n,m}^*)}$,

$$\begin{aligned} x_{n,m} - x^\dagger &= A_{n,m}^\dagger A x^\dagger - x^\dagger = (A_{n,m}^\dagger Q_m A P_n - I)x^\dagger + A_{n,m}^\dagger Q_m A(I - P_n)x^\dagger \\ &= -(I - \Pi_{R(A_{n,m}^*)})x^\dagger + A_{n,m}^\dagger A(I - P_n)x^\dagger. \end{aligned}$$

Thus we have

$$\|x_{n,m} - x^\dagger\| \leq \inf_{z \in R(A_{n,m}^*)} \|x^\dagger - z\| + \|A_{n,m}^\dagger A(I - P_n)x^\dagger\|$$

Now, Lemma 9 and (18) yields the result \square

Remark 2. *In the above result we can easily replace $A_{n,m}^\dagger A(I - P_n)x^\dagger$ by the expression $A_{n,m}^\dagger Q_m A \Pi_{N(Q_m A)^\perp}(I - P_n)x^\dagger$ or by $A_{n,m}^\dagger Q_m A \Pi_{N(Q_m A)^\perp \cap X_n^\perp} x^\dagger$.*

We obtain the first (well-known) result on global convergence; cf. Theorem 1.

Theorem 15. *The approximations $x_{n,m}$ converge to x^\dagger for all $x^\dagger \in N(A)^\perp$ if and only if there exists a constant C such that*

$$\sup_{n,m} \|A_{n,m}^\dagger A\| \leq C. \quad (25)$$

Equivalent to (25) is that there exists a constant C' such that

$$\sup_{n,m} \|A_{n,m}^\dagger A(I - P_n)\| \leq C. \quad (26)$$

Proof. Let $x_{n,m} \rightarrow x^\dagger$ for all $x^\dagger \in N(A)^\perp$ as $m, n \rightarrow \infty$. Then we have by $x_{n,m} = A_{n,m}^\dagger A x^\dagger$ that $A_{n,m}^\dagger A \rightarrow I$ pointwise on $N(A)^\perp$. By the uniform boundedness principle this implies that $A_{n,m}^\dagger A|_{N(A)^\perp}$ must be uniformly bounded. But it is easy to see that this is equivalent to (25).

Conversely let (25) hold, then

$$\|A_{n,m}^\dagger A(I - P_n)x^\dagger\| \leq \|A_{n,m}^\dagger A\| \|(I - P_n)x^\dagger\|,$$

and by (25) and (2), the first result follows. Since $A_n^\dagger A(I - P_n) = A_{n,m}^\dagger A - A_{n,m}^\dagger A_{n,m}$, and $A_{n,m}^\dagger A_{n,m}$ is always bounded, (26) follows. \square

Next, we study condition (25) in depth and rewrite it in other forms. Clearly it holds that

$$\sup_{m,n} \|A_{n,m}^\dagger A\| \leq C \Leftrightarrow \sup_{m,n} \|A^*(A_{n,m}^*)^\dagger\| \leq C. \quad (27)$$

We show that (25) is equivalent to the fact that angles between certain subspaces are uniformly bounded. More precisely, we consider the norm of the product of orthogonal projectors onto two subspaces, which is related to the (minimal canonical) angle; cf. [33, Lemma 5.1].

Lemma 16. *For a sequence of closed subspaces X_n, Y_n , let Π_{X_n}, Π_{Y_n} be the corresponding orthogonal projectors.*

Then we have the following equivalent conditions

$$\exists \rho < 1 : \sup_n \|\Pi_{X_n} \Pi_{Y_n}\| \leq \rho, \quad (28)$$

$$\Leftrightarrow \exists \tau > 0 \forall x \in X_n, y \in Y_n : \|(x+y)\|^2 \geq \tau \|x\|^2, \quad (29)$$

$$\Leftrightarrow \exists \tau' > 0 \forall x \in X_n, y \in Y_n : \|(x+y)\|^2 \geq \tau \|y\|^2. \quad (30)$$

Proof. If (28) holds, then by Young's inequality for any $\epsilon > 0$ $x \in X_n, y \in Y_n$,

$$\|(x+y)\|^2 \geq \|x\|^2 + \|y\|^2 - 2\rho\|x\|\|y\| \geq \|x\|^2(1 - \rho\epsilon) + \|y\|^2(1 - \frac{\rho}{\epsilon}).$$

With $\epsilon = \rho$ or $\epsilon = \rho^{-1}$ either of (30) or (29) follows. Conversely let (29) hold, then then for any $x \in X_n, y \in Y_n$ with $\|x\| = \|y\| = 1$ and any $\epsilon > 0$

$$\epsilon^2 \tau^2 \leq \|\epsilon x + \frac{1}{\epsilon} y\|^2 = \epsilon^2 + \frac{1}{\epsilon^2} - 2(x, y).$$

Taking $\epsilon^2 = (1 - \tau^2)^{-\frac{1}{2}}$ gives the bound

$$2(x, y) \leq 2(1 - \tau^2)^{\frac{1}{2}} < 2,$$

thus with (cf. [33, Lemma 5.1])

$$\|\Pi_X \Pi_Y\| = \sup_{\|x\| \leq 1, \|y\| \leq 1, x \in X, y \in Y} (x, y),$$

the result follows. \square

Combining Proposition 11 and Lemma 16 yields the following

Theorem 17. *The uniform boundedness condition (25) is equivalent to one of the following (and hence all) conditions:*

$$\exists \eta < 1 : \forall n : \quad \|\Pi_{N(A_{n,m}^* A)} \Pi_{R(A_{n,m}^* A)}\| < \eta, \quad (31)$$

$$\exists \eta < 1 : \forall n : \quad \|\Pi_{N(A^* A_{n,m})} \Pi_{R(A^* A_{n,m})}\| < \eta, \quad (32)$$

$$\exists \eta < 1 : \forall n : \quad \|(I - P_n) \Pi_{R(A^* A_{n,m})}\| < \eta. \quad (33)$$

Proof. The boundedness condition can be rephrased as the condition that a constant C exists with (using the notation in Proposition 11)

$$\|v_{n,m}\| \leq C\|x\| = C\|v_{n,m} + u_{n,m}\|.$$

for all $v_{n,m} \in R(A_{n,m}^*A)$ and $u_{n,m} \in N(A_{n,m}^*A)$. However, this is (29) with the spaces $N(A_{n,m}^*A)$ and $R(A_{n,m}^*A)$, thus Lemma 16 gives (31). By (27) we have the equivalent characterization of uniform boundedness using (29) that a constant exists, such that

$$\|\bar{v}_{n,m}\| \leq C\|x\| = C\|\bar{v}_{n,m} + \bar{u}_{n,m}\|,$$

which yields (32). By (30), this is equivalent to the existence of a constant such that

$$\|\bar{w}_{n,m} + \bar{q}_{n,m}\| \leq C\|x\|, \quad (34)$$

with $\bar{w}_{n,m}$, $\bar{q}_{n,m}$ as in Proposition 11. However, $\bar{w}_{n,m}$ is always uniformly bounded for bounded x by (22), and it is orthogonal to $q_{n,m}$. Thus, this condition is satisfied if and only if

$$\|\bar{q}_{n,m}\|^2 \leq C'\|x\|^2 = C'(\|\bar{v}_{n,m} + \bar{q}_{n,m} + \bar{w}_{n,m}\|^2) = C'(\|\bar{v}_{n,m} + \bar{q}_{n,m}\|^2 + \|\bar{w}_{n,m}\|^2).$$

Since we can take $x = \bar{v}_{n,m} + \bar{q}_{n,m} + \bar{w}_{n,m}$ with arbitrary chosen elements $\bar{v}_{n,m}$, $\bar{q}_{n,m}$, $\bar{w}_{n,m}$ out of the corresponding spaces, we have that (34) holds if and only if for all $\bar{v}_{n,m} \in R(A^*A_{n,m})$ and all $\bar{q}_{n,m} \in X_n^\perp$

$$\|q_{n,m}\| \leq C'(\|\bar{v}_{n,m} + \bar{q}_{n,m}\|),$$

which is equivalent to (33). \square

Remark 3. We remark that for closed subspaces X , Y , in Hilbert spaces the identity $\|\Pi_X \Pi_Y\| < 1 \Leftrightarrow \|\Pi_{X^\perp} \Pi_{Y^\perp}\| < 1$ usually does not hold [2, 4].

These conditions can be rewritten in more convenient form.

Theorem 18. The uniform boundedness condition (25) is equivalent to one (and hence all) of the following conditions:

$$\exists C > 0 : \forall n, m, x \quad \|(I - P_n)A^*Q_mAP_nx\| \leq C'\|P_nA^*Q_mAP_nx\|, \quad (35)$$

$$\exists \eta < 1 : \forall n, m, x \quad \|(I - P_n)A^*Q_mAP_nx\| \leq \eta\|A^*Q_mAP_nx\|, \quad (36)$$

$$\exists \eta < 1 : \forall n, m, w \quad \inf_v \|P_nA^*Q_mAw - A^*Q_mAP_nv\| \leq \eta\|P_nA^*Q_mAw\|, \quad (37)$$

$$\exists \eta < 1 : \forall n, m, v \quad \inf_w \|P_nA^*Q_mAw - A^*Q_mAP_nv\| \leq \eta\|A^*Q_mAP_nv\|. \quad (38)$$

Proof. Condition (33) can be rewritten as (36). By splitting the terms using the complementary orthogonal projectors P and $I - P$, it is easy to see that this is equivalent to (35). The identities (37) and (38) are (31) and (32), respectively, when writing the projectors onto the nullspaces as complementary projectors onto the ranges of the adjoints and using the minimization property of such orthogonal projectors. \square

It is not difficult to verify that (35) is equivalent to Vainikko and Hämarik's condition (8). Note that a characterization over angles of subspaces has also been used by Du and Du [6, Theorem 1.2] for the case $Q_m = I$ and with different spaces, which do not yield an equivalent condition to (6) but need additionally the space condition (9).

3.1.1 Necessary and sufficient conditions for convergence

In this section we investigate practically useful conditions such that (25) is satisfied and necessary conditions for (25).

We find a condition of Natterer's type that is equivalent to the uniform boundedness condition extending Natterer's result to the cases of non-injective operators and $Q_m \neq I$.

Proposition 19. *The uniform boundedness condition (25) and hence $x_{n,m} \rightarrow x^\dagger$ for all $x^\dagger \in N(A)^\perp$ holds if and only if there exists a constant C such that for all $x^\dagger \in X$ there exists a $u_n \in X_n$ such that*

$$\|x^\dagger - u_n\| + \|A_{n,m}^\dagger A(x^\dagger - u_n)\| \leq C\|x^\dagger\|. \quad (39)$$

In particular for the case $Q_m = I$, if Natterer's condition (7) holds for all $x^\dagger \in N(A)^\perp$, then (25) holds, thus $x_n \rightarrow x^\dagger$ for all $x^\dagger \in N(A)^\perp$.

Proof. If (25) and hence global convergence holds, then $u_n = x_{n,m}$ satisfies (39). Conversely if (39) holds, then since $Q_m A u_n = A_{n,m} u_n$

$$\begin{aligned} \|A_{n,m}^\dagger A x\| &\leq \|A_{n,m}^\dagger Q_m A(x - u_n)\| + \|A_{n,m}^\dagger A_{n,m} u_n\| \\ &\leq \|A_{n,m}^\dagger A(x - u_n)\| + c_1 \|u_n\| \\ &\leq \|A_{n,m}^\dagger A(x - u_n)\| + c_1 \|u_n - x^\dagger\| + c_1 \|x^\dagger\| \leq C' \|x^\dagger\|. \end{aligned}$$

It is easy to see that (7) implies (39). \square

From the conditions (35)–(38), probably (35) is the most useful. We introduce the norm of the pseudoinverse of the discretized forward operator

$$\|A_{n,m}^\dagger\| = \frac{1}{\sigma_{\min}(Q_m A P_n)} = \sup_{x \in P_n, Q_m A P_n x \neq 0} \frac{(x, x)}{(x, P_n A^* Q_m A P_n x)},$$

where σ_{\min} denotes the smallest (by definition nonzero) singular value. We have the following result:

Lemma 20. *If there exists a constant C such that*

$$\forall n, m \quad \|(I - P_n) A^* Q_m\| \|A_{n,m}^\dagger\| \leq C, \quad (40)$$

then (35) and hence (25) is satisfied.

Proof. In view of (35), we observe that

$$\|P_n A^* Q_m z\| \geq \sigma_{\min}(Q_m A P_n) \|z\|.$$

Taking $z = A P_n x$ and $\|(I - P_n) A^* Q_m A P_n x\| \leq \|(I - P_n) A^* Q_m\| \|Q_m A P_n x\|$ proves the assertion. \square

Note that this result implies in particular Theorem 4. In the same way we could prove the result by replacing (40) by

$$\forall n, m \quad \|(I - P_n) A^* Q_m A\| \|A_{n,m}^\dagger\|^2 \leq C.$$

Natterer [27] has outlined how to prove conditions like (39) in practical situations, namely from inverse inequalities of approximation spaces combined with error estimates for the approximation. Using (40) we can do a similar thing.

Proposition 21. *Let $(H_s, \|x\|_s)_{s \in \mathbb{R}}$ be a Hilbert scale generated by a densely defined unbounded selfadjoint strictly positive operator L , i.e.,*

$$\|x\|_s = \|L^s x\|.$$

Suppose that $A : H_0 \rightarrow H_0$ is such that for some numbers $l, r > 0$

$$c_1 \|x\|_{-l} \leq \|Ax\|_{L^2} \leq c_2 \|x\|_{-r} \quad \forall x \in H_0, \quad (41)$$

and that X_n is a discrete subspace satisfying the approximation condition

$$\|(I - P_n)z\| \leq \gamma_n \|z\|_l \quad \forall z \in H_l,$$

and the inverse inequality

$$\|z_n\|_{L^2} \leq \frac{1}{\beta_n} \|z_n\|_{-r} \quad \forall z_n \in X_n$$

holds. Then if

$$\limsup_n \frac{\gamma_n}{\beta_n} \leq C,$$

the uniform boundedness condition (25) holds.

Proof. We have that

$$\|AP_n x\| \geq c_1 \|P_n x\|_{-l} \geq c_1 \beta_n \|P_n x\|.$$

Thus,

$$\|A_n^\dagger\| \leq \frac{1}{c_1 \beta_n}.$$

Moreover with $z = A^* x$ we find that

$$\|(I - P_n)A^* x\| \leq \gamma_n \|A^* x\|_r.$$

From the right hand side of (41), we see that AL^r is a bounded linear operator and so is its adjoint $L^r A^*$, i.e., $\|A^* x\|_r \leq C \|x\|$. Thus (40) is satisfied by

$$\|(I - P_n)A^* x\| \|A_n^\dagger\| \leq C \frac{\gamma_n}{\beta_n} \leq C.$$

□

In a typical case of finite-element spaces or spline spaces and if we consider a Hilbert scale of Sobolev spaces, then the inverse inequality is usually satisfied with $\beta_n = \frac{1}{n^l}$ and the approximation condition with $\gamma_n = \frac{1}{n^r}$. Thus if $r = l$ is applicable, then we obtain convergence. A similar argument has been utilized by Natterer using condition (7).

The next result concerns the dual variant of (33).

Proposition 22. *If*

$$\exists : \eta < 1 : \forall n : \quad \|P_n \Pi_{N(A_{n,m}^* A)}\| < \eta, \quad (42)$$

then (25) is satisfied. Moreover, (42) holds if

$$\exists : \eta < 1 : \forall n, \forall w : \quad \inf_v \|A^* Q_m A P_n v - P_n w\| \leq \eta \|P_n w\|. \quad (43)$$

Proof. Inequality (42) can be written as

$$\eta > \|\Pi_{N(A_{n,m}^*A)}P_n\| = \|(I - \Pi_{R(A^*A_{n,m})})P_n\|.$$

By the characterization of orthogonal projectors as minimizers we have that this is equivalent to

$$\inf_{v \in R(A^*A_{n,m})} \|v - P_n x\| \leq \eta \|P_n x\|,$$

which is exactly (43). Setting $x = A^*Q_m A w$ we obtain that this implies (37). \square

Note that (42) is not equivalent to (25) because (42) can only hold if the intersection of the corresponding spaces is empty. However, if $X_n \cap N(A_{n,m}^*A) \neq \emptyset$, then (42) cannot hold but (25) still can.

Let us now come to a necessary condition for uniform boundedness. We show that the uniform boundedness (25) implies the space condition (9). In the case $Q_m = I$, this has already been observed by Du [5].

Proposition 23. *Let (25) hold, then*

$$\bigcap_n (N(A) \cap X_n)^\perp = N(A)^\perp,$$

i.e., the space condition (9) holds.

Proof. Since $\bigcup_n (N(A) \cap X_n) \subset N(A)$ it follows that $N(A)^\perp \subset \bigcap_n (N(A) \cap X_n)^\perp$. Thus, we only need to proof the opposite inclusion. Let $x \in \bigcap_n (N(A) \cap X_n)^\perp$. In view of (17) we have that for all n ,

$$\lim_{m \rightarrow \infty} \|x - \Pi_{(N(Q_m A) \cap X_n)^\perp} x\| \rightarrow 0,$$

thus using (21) for x , we have that

$$\forall n : \quad \lim_{m \rightarrow \infty} \overline{w}_{n,m} = 0.$$

By (16), we have that the double sequence $\|\overline{w}_{n,m}\|$ is decreasing in m for all n . Since

$$\|x\|^2 = \|\overline{w}_{n,m}\|^2 + \|\overline{v}_{n,m} + \overline{q}_{n,m}\|^2,$$

we have that for all n , $\|\overline{v}_{n,m} + \overline{q}_{n,m}\|$ is increasing in m and that $\lim_{m \rightarrow \infty} \overline{v}_{n,m} + \overline{q}_{n,m} = x^\dagger$. Thus for all n , $\sup_m \|\overline{v}_{n,m} + \overline{q}_{n,m}\| = \|x^\dagger\|$, and hence $\|\overline{v}_{n,m} + \overline{q}_{n,m}\|$ is bounded uniformly in n, m . Since (25) implies (33) using (30), we have a constant C such that

$$\|\overline{q}_{n,m}\| \leq C \|\overline{v}_{n,m} + \overline{q}_{n,m}\| \leq C \|x\|.$$

Thus, $\|\overline{q}_{n,m}\|$ is uniformly bounded, it has a weakly convergent subsequence as $n, m \rightarrow \infty$, and as $q_{n,m} \in X_n^\perp$, it follows that this limit can only be 0. By a subsequence argument we conclude that $\lim_{n,m \rightarrow \infty} \overline{q}_{n,m} = 0$. It follows that the iterated limit $\lim_{n \rightarrow \infty} (\lim_{m \rightarrow \infty} \overline{q}_{n,m}) = 0$. Thus,

$$x = \lim_{n \rightarrow \infty} \left(\lim_{m \rightarrow \infty} \overline{v}_{n,m} \right).$$

Since each $v_{n,m}$ is in $N(A)^\perp$, and this space is weakly closed, all the limits are in $N(A)^\perp$ as well, thus $x \in N(A)^\perp$. \square

3.2 Conditions for local convergence

We are now interested in local convergence results, i.e., to study the question if for a given a specific element $x^\dagger \in N(A)^\perp$ the corresponding sequence $x_{n,m}$ converges (weakly or strongly). The difference to the previous section is that the conditions imposed here are not “uniform” in x^\dagger but depend on the specific x^\dagger .

A practically useful sufficient condition for strong convergence is a simple consequence of Lemma 14 (compare Theorem 4).

Proposition 24. *If*

$$\lim_{m,n \rightarrow \infty} \|A_{n,m}^\dagger\| \|A(I - P_n)x^\dagger\| \rightarrow 0,$$

then $x_{n,m} \rightarrow x^\dagger$ as $m, n \rightarrow \infty$.

Hence if x^\dagger can be approximated well in X_n , we can hope for strong convergence. This result is quite crude compared to Theorem 5, where (except for weak convergence) equivalent conditions to convergence are established. However, the mentioned theorem of Groetsch and Neubauer (with the supplementary result of Du) can also be extended with minor modifications to the case of $Q_m \neq I$.

Theorem 25. *We have the following local convergence conditions for strong convergence.*

$$x_{n,m} \rightarrow x^\dagger \iff \limsup_{n,m} \|x_{n,m}\| \leq \|x^\dagger\|. \quad (44)$$

Suppose that the space condition (9) holds. Then we have the following local convergence conditions for weak convergence.

$$x_{n,m} \rightharpoonup x^\dagger \iff \sup_{n,m} \|x_{n,m}\| < \infty.$$

Proof. Consider first the part on weak convergence. By boundedness, $x_{n,m}$ has a weakly convergent subsequence with limit u . As in [9] it follows immediately that $u - x^\dagger \in N(A)$. Moreover each $x_{n,m} \in (N(Q_m A) \cap X_n)^\perp \subset (N(A) \cap X_n)^\perp$, thus $x_{n,m} \in (N(A) \cap X_k)^\perp$ for all $k \geq n$. This implies that $u \in \bigcap_n (N(A) \cap X_n)^\perp$, and by (9), $u \in N(A)^\perp$. Thus $u - x^\dagger \in N(A) \cap N(A)^\perp$, hence $u = x^\dagger$. By a subsequence argument, $x_{n,m} \rightharpoonup x^\dagger$. For (44), we do not need (9). The proof follows [5]: from (44), we again find a weakly convergence subsequence with limit u and $u - x^\dagger \in N(A)$ and $x^\dagger \in N(A)^\perp$. Thus,

$$\|u - x^\dagger\|^2 + \|x^\dagger\|^2 = \|u\|^2 \leq \liminf_{n,m} \|x_{n,m}\|^2 \leq \limsup_{n,m} \|x_{n,m}\|^2 \leq \|x^\dagger\|^2,$$

thus $u = x^\dagger$ and as before $x_{n,m} \rightharpoonup x^\dagger$. From (44) we also find that $\|x_{n,m}\| \rightarrow \|x^\dagger\|$, which together with weak convergence implies strong convergence. The other directions of the implications are trivial. \square

In a next step, we replace (9) by other “local” conditions.

Lemma 26. *We have that*

$$x_{n,m} \rightharpoonup x^\dagger \iff \begin{cases} \sup_{n,m} \|x_{n,m}\| < \infty & \text{and} \\ \Pi_{N(A)} x_{n,m} \rightharpoonup 0. \end{cases} \quad (45)$$

Proof. Suppose that $x_{n,m}$ converges weakly to x^\dagger . Then for arbitrary z

$$\begin{aligned}\lim_{n,m \rightarrow \infty} (\Pi_{N(A)} x_{n,m}, z) &= \lim_{n,m \rightarrow \infty} (x_{n,m}, \Pi_{N(A)} z) \\ &= \lim_{n,m \rightarrow \infty} (x_{n,m} - x^\dagger, \Pi_{N(A)} z) = 0,\end{aligned}$$

where we used that $(x^\dagger, \Pi_{N(A)} z) = 0$ as $x^\dagger \in N(A)^\perp$. Conversely, let $x_{n,m}$ be bounded and $\Pi_{N(A)} x_{n,m} \rightharpoonup 0$. As in the proof before, $x_{n,m}$ has a weakly convergent subsequence with limit u such that $u - x^\dagger \in N(A)$. Thus with $z = u - x^\dagger$ we have that

$$\begin{aligned}0 &= \lim_k (\Pi_{N(A)} x_{n_k, m_k}, u - x^\dagger) = \lim_k (x_{n_k, m_k}, \Pi_{N(A)} (u - x^\dagger)) \\ &= \lim_k (x_{n_k, m_k} - x^\dagger, \Pi_{N(A)} (u - x^\dagger)) = \|u - x^\dagger\|^2,\end{aligned}$$

thus $u = x^\dagger$. By a subsequence argument $x_{n,m} \rightharpoonup x^\dagger$. \square

As a consequence, we can find sufficient conditions for weak and strong local convergence generalizing the results of Luecke and Hickey.

Proposition 27. *Suppose that with some fixed constant C , there exists for each n, m an index pair $(\tilde{m}_{n,m}, \tilde{n}_{n,m}) \in \mathbb{N} \times \mathbb{N}$ with $\lim_{n,m \rightarrow \infty} \tilde{n}_{n,m} = \infty$, and $\tilde{m}_{n,m} \geq m$, and $\tilde{n}_{n,m} \leq n$, such that*

$$\sup_{n,m} \|x_{n,m}\| < C \quad \text{and} \quad \sup_{n,m} \|A^*(A_{\tilde{m}_{n,m}, \tilde{n}_{n,m}}^*)^\dagger x_{n,m}\| < C. \quad (46)$$

Then $x_{n,m} \rightharpoonup x^\dagger$ as $m, n \rightarrow \infty$.

If we can choose $(\tilde{m}_{n,m}, \tilde{n}_{n,m}) = (n, m)$, i.e.,

$$\sup_{n,m} \|x_{n,m}\| < C \quad \text{and} \quad \sup_{n,m} \|A^*(A_{n,m}^*)^\dagger x_{n,m}\| < C, \quad (47)$$

then $x_{n,m} \rightarrow x^\dagger$ as $m, n \rightarrow \infty$.

Proof. We apply (21) and get for all n, m

$$x_{n,m} = A^*(A_{\tilde{m}_{n,m}, \tilde{n}_{n,m}}^*)^\dagger x_{n,m} + \overline{w}_{\tilde{m}_{n,m}, \tilde{n}_{n,m}} + \overline{q}_{\tilde{m}_{n,m}, \tilde{n}_{n,m}}.$$

By (46) it follows that $\overline{w}_{\tilde{m}_{n,m}, \tilde{n}_{n,m}} + \overline{q}_{\tilde{m}_{n,m}, \tilde{n}_{n,m}}$ is uniformly bounded, and since these two elements are orthogonal to each other it follows that both components are uniformly bounded as well, hence they have weakly convergent subsequences as $n, m \rightarrow \infty$ with limit w, q .

For fixed k , $P_k q = \lim_{n,m \rightarrow \infty} P_k \overline{q}_{\tilde{m}_{n,m}, \tilde{n}_{n,m}} = 0$ since $\tilde{n}_{n,m} \rightarrow \infty$, and thus it follows that $q = 0$. Since $x_{n,m} \in (N(Q_{\tilde{m}_{n,m}} A) \cap X_n)^\perp$, for $\tilde{m}_{n,m} \geq m$, and $(N(Q_{\tilde{m}_{n,m}} A) \cap X_n)^\perp \subset (N(Q_{\tilde{m}_{n,m}} A) \cap X_{\tilde{n}_{n,m}})^\perp$, we have that $\overline{w}_{\tilde{m}_{n,m}, \tilde{n}_{n,m}} = 0$. Thus we have for a subsequence

$$\lim_{k \rightarrow \infty} \left(x_{n_k, m_k} - A^*(A_{\tilde{m}_{m_k, n_k}, \tilde{n}_{m_k, n_k}}^*)^\dagger x_{n_k, m_k} \right) = 0.$$

By a subsequence argument we have that this holds for the whole sequence.

$$\lim_{n,m \rightarrow \infty} (x_{n,m} - A^*(A_{\tilde{m}_{n,m}, \tilde{n}_{n,m}}^*)^\dagger x_{n,m}) = 0. \quad (48)$$

It follows that

$$\Pi_{N(A)}x_n = \Pi_{N(A)}(x_{n,m} - A^*(A_{n,m}^*)^\dagger x_{n,m}) \rightharpoonup 0, \quad \text{as } m, n \rightarrow \infty,$$

thus, by (45) we obtain the result that $x_n \rightharpoonup x^\dagger$ as $m, n \rightarrow \infty$. Since (47) is a special case of (46), we have that under (47), $x_{n,m}$ converges weakly to x^\dagger and furthermore by (48) also that $A^*(A_{n,m}^*)^\dagger x_{n,m} \rightharpoonup x^\dagger$ as $m, n \rightarrow \infty$. Thus, by weak convergence,

$$\lim_{n,m \rightarrow \infty} \|x_{n,m}\|^2 = \lim_{n,m \rightarrow \infty} (A^*(A_{n,m}^*)^\dagger x_{n,m}, x^\dagger) = (x^\dagger, x^\dagger) = \|x^\dagger\|^2.$$

By the Radon-Riesz property, we obtain that $\|x_{n,m} - x^\dagger\| \rightarrow 0$. \square

Remark 4. Proposition 27 includes Theorem 6 as a special case. Indeed, setting $Q_m = I$, from (11), the boundedness of x_n and $\|A^*(A_n^*)^\dagger x_n\|$ follows immediately, and thus by (47) we obtain Theorem 6 as a corollary.

4 A counterexample

In this section we provide a nontrivial example of a sequence of projected least-squares solutions, x_n , which is bounded but non even weakly convergent. Note that Du's example considers a similar situation but the sequence x_n is strongly convergent (but not to x^\dagger). The example again stresses the importance of the space conditions (9) and the fact that the part on weak convergence in Theorem 5 is false without the space condition (9).

Theorem 28. *There exists an operator A and x^\dagger and a sequence of finite-dimensional spaces $(X_n)_n$ satisfying (2) such that $x_{2n} \rightharpoonup u$ but $x_{2n} \not\rightarrow u$ and $x_{2n+1} \rightharpoonup v$, but $x_{2n+1} \not\rightarrow v$ and $u, v \neq x^\dagger$. In particular the sequence x_n neither converges weakly nor strongly and it has no weakly convergent subsequence has limit x^\dagger .*

Proof. Let X be a separable Hilbert space with orthonormal basis e_{ij} , $(i, j) \in \mathbb{N} \times \mathbb{N}$, i.e., all elements $x \in X$ may be represented via

$$x = \sum_{i,j=1}^{\infty} \xi_{ij} e_{ij} \quad \text{with} \quad \|x\|^2 = \sum_{i,j=1}^{\infty} \xi_{ij}^2 < \infty.$$

We define a linear bounded operator $A : X \rightarrow X$ via

$$Ax := \sum_{i=1}^{\infty} \sum_{j=2}^{\infty} (\xi_{ij} + q^j \xi_{i1}) e_{ij},$$

where $q \in (0, 1)$ is fixed. Obviously,

$$\|Ax\|^2 = \sum_{i=1}^{\infty} \sum_{j=2}^{\infty} (\xi_{ij} + q^j \xi_{i1})^2 \leq 2 \max \left\{ 1, \frac{q^4}{1-q^2} \right\} \|x\|^2.$$

It is easy to see that A has an infinite-dimensional nullspace. It holds that

$$x \in N(A) \iff \forall i \geq 1, j > 1 : \xi_{ij} = -q^j \xi_{i1} \quad \text{and} \quad \sum_{i=1}^{\infty} \xi_{i1}^2 < \infty. \quad (49)$$

A generalized solution $x^\dagger = A^\dagger y$ of the equation $Ax = y$ is always an element of $N(A)^\perp$; we may characterize these elements as follows:

$$z = \sum_{i,j=1}^{\infty} \eta_{ij} e_{ij} \in N(A)^\perp \iff \forall i \geq 1 : \begin{matrix} \eta_{i1} = \sum_{j=2}^{\infty} q^j \eta_{ij} \text{ and} \\ \sum_{i=1}^{\infty} \sum_{j=2}^{\infty} \eta_{ij}^2 < \infty \end{matrix} . \quad (50)$$

Now we choose finite-dimensional subspaces of X :

$$X_n := \text{span}\{e_{ij} : 1 \leq i, j \leq n\}.$$

Obviously, (2) holds. Let $x^\dagger \in N(A)^\perp$ with $x^\dagger = \sum_{i,j=1}^{\infty} \zeta_{ij} e_{ij} \in N(A)^\perp$ and $y := Ax^\dagger$, and set $x_n := A_n^\dagger y$, where $A_n := AP_n$.

Since, due to (49), $N(A) \cap X_n = \{0\}$, we get by (14) that $N(A_n) = X_n^\perp$. Therefore,

$$x_n = \sum_{i,j=1}^n \xi_{ij}^n e_{ij} \quad (51)$$

is the unique minimizer in X_n of the problem

$$\|Ax_n - y\|^2 = \sum_{i=1}^{\infty} \sum_{j=2}^{\infty} ((\xi_{ij}^n - \zeta_{ij}) + q^j (\xi_{i1}^n - \zeta_{i1}))^2 \rightarrow \min ,$$

where $\xi_{ij}^n := 0$ if $i > n$ or $j > n$.

From the first order necessary conditions for a minimum we obtain the solution:

$$\xi_{ij}^n = \zeta_{ij} - q^j (\xi_{i1}^n - \zeta_{i1}), \quad \text{if } 1 \leq i \leq n, 2 \leq j \leq n,$$

and

$$\begin{aligned} \xi_{i1}^n &= \zeta_{i1} + \left(\sum_{j=n+1}^{\infty} q^{2j} \right)^{-1} \sum_{j=n+1}^{\infty} q^j \zeta_{ij} \\ &= \zeta_{i1} + (1 - q^2) q^{-(n+1)} \sum_{j=0}^{\infty} q^j \zeta_{i,n+1+j}, \quad 1 \leq i \leq n. \end{aligned}$$

Now we choose a concrete element $x^\dagger \in N(A)^\perp$:

$$\zeta_{ij} := \frac{q^j}{1 - q^2} (c_i \rho_j + r_{ij}), \quad i \geq 1, j > 1,$$

with

$$\rho_j := \begin{cases} 1, & j \text{ even}, \\ 0, & j \text{ odd}, \end{cases} \quad r_{ij} := \begin{cases} 1, & j = i + 1, \\ 0, & \text{else}, \end{cases} \quad \sum_{i=1}^{\infty} c_i^2 < \infty, \quad (52)$$

and the extension (50) for $j = 1$. The condition on the coefficients c_i guarantees that

$$\sum_{i=1}^{\infty} \sum_{j=2}^{\infty} \zeta_{ij}^2 < \infty.$$

It then holds that

$$\xi_{i1}^n = \zeta_{i1} + c_i e_n + r_{i,n+1} \quad \text{with} \quad e_n := \sum_{j=0}^{\infty} q^{2j} \rho_{n+1+j}. \quad (53)$$

Noting that

$$e_n = \frac{1}{1-q^4} \cdot \begin{cases} q^2, & n \text{ even}, \\ 1, & n \text{ odd}, \end{cases} \quad (54)$$

(53) implies that

$$\lim_{l \rightarrow \infty} \xi_{i1}^{2l} = \zeta_{i1} + \frac{c_i q^2}{1-q^4}, \quad (55)$$

$$\lim_{l \rightarrow \infty} \xi_{i1}^{2l+1} = \zeta_{i1} + \frac{c_i}{1-q^4}. \quad (56)$$

Let us now define the two elements

$$u := \sum_{i,j=1}^{\infty} u_{ij} e_{ij} \quad \text{and} \quad v := \sum_{i,j=1}^{\infty} v_{ij} e_{ij}$$

with

$$\begin{aligned} u_{i1} &:= \zeta_{i1} + \frac{c_i q^2}{1-q^4} & u_{ij} &:= \zeta_{ij} - q^j (u_{i1} - \zeta_{i1}), \quad j > 1, \\ v_{i1} &:= \zeta_{i1} + \frac{c_i}{1-q^4} & v_{ij} &:= \zeta_{ij} - q^j (v_{i1} - \zeta_{i1}), \quad j > 1. \end{aligned}$$

Obviously, due to (49) and (52), $u - x^\dagger$ and $v - x^\dagger \in N(A)$, and thus u and v are least-squares solutions of $Ax = y = Ax^\dagger$ with $u, v \neq x^\dagger$. Together with (52), (53), and (54) we immediately obtain that (remember that $x_n = A_n^\dagger y$ is given by (51), (53))

$$\|x_{2l} - P_{2l}u\|^2 = \frac{q^4 - q^{2n+2}}{1-q^2} = \|x_{2l+1} - P_{2l+1}v\|^2. \quad (57)$$

Now (55) and (56) imply that

$$x_{2l} \rightharpoonup u, \quad x_{2l+1} \rightharpoonup v,$$

but (57) implies $x_{2l} \not\rightarrow u$ and $x_{2l+1} \not\rightarrow v$. Thus, it is possible that x_n has different weakly convergent subsequences, but it neither converges weakly nor strongly towards x^\dagger . \square

5 Conclusion

We have studied global and local convergence of general projection schemes for ill-posed problems. For global convergence, we have established the uniform boundedness condition (25) as being necessary and sufficient and have found concrete conditions in Theorems 17 and 18 when this holds. Several practically useful sufficient condition were given in Section 3.1.1. Concerning local convergence, we have generalized the well-known results of Groetsch and Neubauer and

Du giving an equivalent characterization of local convergence by norm bounds in Theorem 25. Further sufficient conditions of the type of Luecke and Hickey were given in Proposition 27.

In the analysis, we point out two important findings: the recognition of the $x_{n,m}$ as oblique projection of x^\dagger , which leads to a study of angles of a sequence of subspaces. The second point is the question if the intuitive identity “ $N(A)^\perp = \lim_{m,n} N(A_{n,m})^\perp$ ” is valid, understood in the sense as (24). As the inclusion “ \subset ” always holds, this gives a way of applying the uniform boundedness principle. However it is important to notice that this identity does only hold unless the additional space condition (9) holds. While for injective operators this is trivially true, for noninjective operators (9) has to be taken into account when studying local (weak) convergence.

The issue that this condition is needed for weak convergence is illustrated by a nontrivial counterexample in Theorem 28 of a bounded sequence x_n which does not converge at all, thus generalizing the examples of Seidman and Du.

Acknowledgments

The author would like to thank Andreas Neubauer for useful discussions and for providing the counterexample in Theorem 28.

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